STAT 230
TEST 1

KEY

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PART I.

Multiple Choice Questions. Answer the following questions by circling the best possible answer. Exactly one answer must be circled.

1. Which of the following statements is correct?

   (a) If A and B are two events that overlap then they are independent.
   (b) If A and B are dependent, they cannot be disjoint.
   (c) Both of the above statements are correct.
   (d) None of the above statements is correct.

2. Which of the following statements is correct?

   (a) If A and B are independent, B and C are independent, then A and C are independent.
   (b) If A and B are independent, B and C are independent, A and C are independent, then A, B and C are independent.
   (c) Both of the above statements are correct.
   (d) None of the above statements is correct.

3. Which of the following statements is correct?

   (a) The sample mean is always equal to the mean of the empirical distribution.
   (b) The sample variance is always equal to the variance of the empirical distribution.
   (c) Both of the above statements are correct.
   (d) None of the above statements is correct.

4. Which of the following random variables is not based on repeating independent Bernoulli trials?

   (a) a geometric random variable
   (b) a Poisson random variable
   (c) a negative binomial random variable
   (d) all of the above are based on repeating independent Bernoulli trials

5. A random variable X with distribution $f(x) = \dfrac{15}{4x!(5-x)!}$, $x = 0, 1, \cdots, 5$. The variance of X is

(a) $\frac{15}{2}$
(b) $\frac{15}{4}$
(c) $\frac{5}{2}$
(d) $\frac{5}{4}$

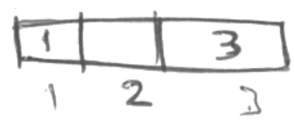
6. A box has two cards that are identical in shape and texture. The first card has a red face and a green face while the second card has both faces red. One card is selected at random and noted to have a red face. The probability that the other face of the card selected is also red is

(a) $\frac{1}{2}$
(b) $\frac{1}{3}$
(c) $\frac{2}{3}$
(d) Not enough information, because the red side could be any face of the second card

7. n balls numbered $1, \cdots, n$ are thrown randomly in n boxes numbered $1, \cdots, n$. We say that a match occurs in box i if the ball labeled i lands in the i^{th} box. The probability of exactly $n-1$ matches is

(a) 0
(b) 1
(c) $\frac{n-1}{n}$
(d) not knowable without knowing n

$n=3$



exactly 2 is possible?

no

because the 3rd then has to be a match!

8. Three identical balls are thrown randomly into five identical boxes. The probability that at least one box receives two or more balls is

   (a) 0.48
   (b) 0.52
   (c) 0.10
   (d) 0.60

9. An absent minded professor has four keys in his pocket. Exactly one key opens his office door. He tries the keys one by one until he opens his door. Every time he tries a wrong key he discards it. The expected number of keys that he has to try until he opens his door (including the right key) is

   (a) 4
   (b) 5
   (c) $\frac{5}{4}$
   (d) $\frac{10}{4}$

K	1	2	3	4
p(K)	$\frac{1}{4}$	$\frac{3}{4}\cdot\frac{1}{3}$ $=1/4$	$\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}$ $=1/4$	$\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}$ $=1/4$

$$\frac{\frac{4(5)}{2}}{4} = \frac{5}{2}$$

P[3 keys tried]

= P.[first is W & second is W & 3rd R]

$= P[W_1]\, P[W_2 \mid W_1] . P[R_3 \mid W_1 W_2]$

10. A recurrence relation for the number s_n of binary strings of size $n > 2$ that end with '111' and that do not have '111' anywhere else is

   (a) $s_n = s_{n-1} + s_{n-2}$
   (b) $s_n = 2s_{n-1} + s_{n-2}$
   (c) $s_n = s_{n-1} + 2s_{n-2}$
   (d) $s_n = s_{n-1} + s_{n-2} + s_{n-3}$
   (e) $s_n = s_{n-1} + s_{n-2} + 1$

11. An experiment consists of repeatedly drawing chips from a box that contains three white chips and three black chips. Let A be the event that the fourth item selected is the third black chip provided that chips are selected with replacement. Let B be the same event but when the experiment is done without replacement. Which of the following is correct?

$$P(A) = \binom{4-1}{3-1} \cdot \left(\tfrac{1}{2}\right)^4 = \frac{3}{16} \qquad p = \tfrac{1}{2}$$

neg. binom.

(a) $P(A) = P(B)$
(b) $P(A) > P(B)$
(c) $P(A) < P(B)$

$P(B) = P(B_1 B_2)$ where B_1 : 2 of the first 3 are Black, B_2 : 4th is Black

B_2 : 1 Black 2 White

$$P(B_1) \cdot P(B_2 | B_1) = \frac{\binom{3}{2}\binom{3}{1}}{\binom{6}{3}} \cdot \frac{1}{3} = \frac{3}{\frac{6\times5\times4}{3\times2}} = \boxed{\frac{3}{20}}$$

12. The number of three-letter codes with letters chosen from the set of letters in the word "conquer" is

(a) 7!
(b) 3!
(c) $\frac{7!}{3!}$
(d) $\frac{7!}{4!}$

13. The moment generating function of a certain random variable X is given as $M(t) = \dfrac{(e^t+1)^5}{32}$. The probability $P(X = 0)$ is

(a) $\frac{1}{2}$
(b) $\frac{1}{32}$
(c) $\frac{1}{4}$
(d) $\frac{1}{5}$

14. Let A be the event that we observe three heads upon tossing a fair coin ~~ten~~ times. Let B be the event that we observe four heads upon tossing a fair coin eight times. Which of the following is correct?

(a) $P(A) = P(B)$ because 3 and 4 are the expected numbers of heads respectively.
(b) $P(A) > P(B)$
(c) $P(A) < P(B)$

$$\frac{\binom{10}{3}}{2^{10}} < \frac{\binom{8}{4}}{2^8}$$

$$\frac{10\times9\times8}{6\cdot2^{10}} < \frac{8\times7\times6\times5}{4\times3\times2}$$

$$30 \quad \frac{10\times3\times4}{2^8} < \frac{}{2^8} = \frac{70}{2^8}$$

PART II.

Free Response Questions. Write detailed solutions to the following questions and justify your answers.

1. Box 1 contains five white chips and five black chips. Box 2 contains four white chips and four black chips. Two chips are selected from box 1 and placed in box 2 without noticing their color. Next a chip is selected from box 2 and ~~found to be white~~. What is the probability that at least one white chip was selected in the first set of two chips?

found to be white

Let A: at least 1 w. in box 1

B: 1 w from box 2.

$$P(A|B) = \frac{P(A)\cdot P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

$$P[A] = P[1\text{ chip}] + P[2\text{ chips}] = \frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}} + \frac{\binom{5}{2}}{\binom{10}{2}} = \frac{25}{\frac{10\times 9}{2}} + \frac{5\times 4}{10\times 9} = \boxed{\frac{70}{90}}$$

$$P[B|A] = P[\text{exactly } 1\text{ w} \mid A]\cdot P[B \mid A \,\&\, \text{exactly white}] + P[\text{exactly } 2\text{ white}]\cdot P[B \mid A \,\&\, \text{exactly two}]$$

$$= \frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}}\cdot\frac{5}{10} + \frac{\binom{5}{2}}{\binom{10}{2}}\cdot\frac{6}{10} = \frac{50}{90}\cdot\frac{5}{10} + \frac{20}{90}\cdot\frac{6}{10} = \boxed{\frac{37}{90}}$$

$$P[B|A^c] = \frac{4}{10};\quad P[A^c] = \frac{\binom{5}{2}}{\binom{10}{2}}.$$

$$P[A|B] = \frac{\frac{70}{90}\cdot\left[\frac{37}{90}\right]}{\frac{70}{90}\cdot\frac{37}{90} + \frac{20}{90}\cdot}$$

$$P[B|A] = P[\text{exactly one white}|A] \cdot P[B|\text{exactly one } w] + P[\text{exactly 2 white}|A] \cdot P[B|\text{exactly 2 white}]$$

$$= \frac{1}{P(A)}\left[P[\text{exactly one } w] \cdot P[B|\text{ex. one } w] + P[2W]\,P[B|2W]\right]$$

$$= \frac{1}{\frac{70}{90}}\left[\frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}} \cdot \frac{5}{10} + \frac{\binom{5}{2}}{\binom{10}{2}} \cdot \frac{6}{10}\right] = \frac{37/90}{70/90} = \boxed{\frac{37}{70}}$$

$$P[B|A^c] = \frac{4}{10}.$$

$$\text{So } P[A|B] = \frac{P[A] \cdot P[B|A]}{P[A] \cdot P[B|A] + P[A^c]\,P[B|A^c]}$$

$$= \frac{P[W_1]\,P[B|W_1] + P[W_2]\,P[B|W_2]}{P[W_1]P[B|W_1] + P[W_2]P[B|W_2] + P[A^c]\,P[B|A^c]}$$

$$= \frac{\binom{5}{1}\binom{5}{1} \cdot \frac{5}{10} + \binom{5}{2} \cdot \frac{6}{10}}{\binom{5}{1}\binom{5}{1} \cdot \frac{5}{\cancel{10}} + \binom{5}{2}\frac{6}{\cancel{10}} + \binom{5}{2} \cdot \frac{4}{\cancel{10}}} = \frac{125 + 60}{125 + 60 + 40}$$

$$\boxed{= \frac{185}{225}} = 0.84.$$

$$\text{Compare with } P[A] = \frac{7}{9} \approx 0.77$$

2. A box has 10 white and 10 black chips. Two chips are removed and their color is noted. If they are not both white they are replaced in the box and then the same experiment is tried again. Let X be the number of times we need to try until we get both chips to be white. Derive the moment generating function of X and hence find $E[X]$.

It is evident, because of replacement, that X is a geometric random variable with $p = \text{Prob}(\text{both chips white})$

$$= \frac{\binom{10}{2}}{\binom{20}{2}} = \frac{\frac{10\times 9}{2}}{\frac{20\times 19}{2}} = \frac{9}{38}$$

$$\text{so } M(t) = \sum_{x=1}^{\infty} e^{tx} p \cdot q^{x-1} = pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= pe^t \sum_{K=0}^{\infty} (qe^t)^K = \frac{pe^t}{1-qe^t} \; ; \text{ where } qe^t < 1$$

$$\Leftrightarrow t < -\ln q.$$

$$M'(t) = \frac{pe^t(1-qe^t) + qe^t \cdot pe^t}{(1-qe^t)^2}$$

$$\text{So } EX = M'(0) = \frac{p(1-q) + pq}{(1-q)^2} = \frac{p[1-q+q]}{p^2} = \frac{1}{p}.$$

$$= \frac{38}{9}$$

Grading:

$\frac{1}{2}$ pt: getting ~~geom~~ prob. 2 white.

1 pt.: getting that it is geometric

$\frac{1}{2}$ pt.: derive $M(t)$

$\frac{1}{2}$ pt.: $M'(t)$

$\frac{1}{2}$ pt.: $M'(0) = \frac{1}{p} = \frac{38}{9}$

bonus How many arrangements of the word × × × ○ ○ ○ ○○ has at least two consecutive ×s?

if no condition: $\binom{8}{3}$

To get a string with no two consecutive X's $\wedge O \wedge O \wedge O \wedge O \wedge O \wedge$

use the Os as separators & choose 3 spots out of the 6 locations: $\binom{6}{3}$

Answer is total - complement $= \binom{8}{3} - \binom{6}{3} =$

$\frac{8 \times 7 \times \cancel{6}}{\cancel{3 \times 2}} - \frac{6 \times 5 \times 4}{\cancel{3 \times 2}} = 56 - 20 = \boxed{36}$

OR: directly Two cases: the X's are

1) all together

6 possible places

(2) Two & one

$_6P_2 = 6 \times 5 = 30$

Answer is $6 + 30 = 36$